A GAMES PROBLEM OF CORRECTION OF MOTION

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The problem of correction of motion for the purpose of minimizing the disparity between the chosen coordinates at a given instant ϑ is considered. The problem is complicated by the lack of complete information on the present phase state $x[t]$ of the controlled object. This produces a new games situation interpreted as a differential game concerning two converging motions. The equations of motion are considered in linear approximation. The present paper is closely related in subject-matter to papers [1–4] et al.

1. Formulation of the problem. Let us consider the motion $x[t]$ described in linear approximation by the equation

$$dx/dt = A(t)\,x + B(t)\,u \tag{1.1}$$

Here $x = \{x_i\}$ is the n-dimensional phase vector of the object measured from the prescribed motion $x^\circ[t] \equiv 0$; the r-dimensional vector u describes the controlling forces. The realizations $u[t]$ of the permissible control u are restricted by the condition

$$u[t] \in U \tag{1.2}$$

where U is some bounded convex closed set in the r-dimensional vector space. We shall consider the process over the segment $0 \leqslant t \leqslant \vartheta$. The deviation of the motion $x[t]$ from the prescribed motion $x^\circ[t] \equiv 0$ is given by the quantity

$$\gamma[t] = \|\{x[t]\}_m\| \tag{1.3}$$

where the symbol $\{x\}_m$ denotes a vector consisting of the first m coordinates of the vector x; the symbol $\|q\|$ denotes the Euclidean norm of the vector q.

The control goal is to minimize $\gamma[\vartheta]$. The problem is complicated by the impossibility of measuring the present values of $x[t]$ exactly. We assume that information about the phase states $x[t]$ $(0 \leqslant t \leqslant \vartheta)$ is supplied [5] by the monitored signal $w[\tau]$ $(0 \leqslant \tau \leqslant t)$ which is related to the phase vector $x[\tau]$ by the expression

$$w[\tau] = P(\tau)\,x[\tau] + f[\tau] \tag{1.4}$$

Here $w[\tau]$ is a k-dimensional vector function and $f[\tau]$ is the measurement error. The realization of $f[\tau]$ is not known, but we do know that

$$\varkappa[f[\tau], t] \leqslant \nu \qquad (0 \leqslant \tau \leqslant t) \tag{1.5}$$

This inequality restricts the intensity ([5], p. 273) of $\varkappa$ possible realizations of $f[\tau]$. We assume that the quantity $\varkappa[f, t]$ is some norm of the vector function $f[\tau]$ $(0 \leqslant \tau \leqslant t)$. We assume, moreover, that the domain $\Gamma[0]$ of possible initial values of the phase vector $x[0]$ has been defined, and that it is possible to measure the values of the generated controlling forces $u[\tau]$.

Our task to find a control u which will deliver a motion $x[t]$ ensuring the smallest possible value of $\gamma[\vartheta]$ (1.3) for the least favorable cases of the error $f[\tau]$. Let us state the problem more precisely.

Let a phase state $x[t]$ unknown to the organs generating the control $u[t]$ be realized at the instant t $(0 \leqslant t < \vartheta)$. On the basis of the known signal $\{w[\tau]$ $(0 \leqslant \tau \leqslant t)$, $u[\tau]$ $(0 \leqslant \tau < t)\}$ it is possible to form a notion of the domain $\Gamma[t]$ of the phase space $\{x\}$ in which $x[t]$ may lie. (Estimation of the domain $\Gamma[t]$ is effected by some operation $\varphi^\circ[\{w, u\}, t]$ which will be described below). By the feedback principle, the control $u[t]$ realized at a given instant t must be chosen on the basis of the estimate of the domain $\Gamma[t]$. It would therefore be convenient to construct the control in the form of a functional relation, i.e. in the form

$$u[t] = u(t, \Gamma[t]) \tag{1.6}$$

However, in order to make our analysis valid for continuous control laws u characteristic of games problems we assume (e.g. see [6]) that the estimate of the domain $\Gamma[t]$ obtained by the instant t defines a whole set $U(t, \Gamma[t])$ of possible values of $u[t]$ rather than just a single value of $u[t]$ (1.6). Thus, the control law for system (1.1) (the strategy U) is described by the contingency

$$u[t] \in U(t, \Gamma[t]) \tag{1.7}$$

Hence, in order to specify some control law u for system (1.1) (i.e. in order to choose some strategy U) we must specify a system of sets $U(t, \Gamma)$ which is defined for all t from the segment $[0, \vartheta)$ and for all possible domains $\Gamma = \Gamma[t]$ occurring during the process. We therefore identify the strategy U with the system of sets $U(t, \Gamma)$. The motion $x[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ of system (1.1) for some strategy U is an absolutely continuous vector function $x[t]$ which satisfies Eq. (1.1) for $u = u[t]$ from (1.7) for almost all values of t from the segment $[t_0, \vartheta]$. We call the strategy U "permissible" if its choice ensures that any initial state $x[t_0] = x_0$ (and $\Gamma[t_0]$) generates a motion $x[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ of system (1.1).

Let the unknown state $x[t_0] = x_0$ and the known estimate $\Gamma[t_0]$ be realized. The chosen permissible strategy U (1.7) and the relization of the error $f[t]$ $(t \geqslant t_0)$ in (1.4) determine the motions $x[t]$ $(t \geqslant t_0)$. We denote the resulting quantity $\gamma[\vartheta]$ (1.3) by $\gamma(\vartheta \mid t_0, \Gamma[t_0], U, f)$. We are now ready to formulate the problem in the following way.

Problem 1.1. We are to find the optimal strategy U° form among the permissible strategies U which ensures fulfilment of the inequality

$$\gamma(\vartheta \mid t_0, \Gamma[t_0], U^\circ, f) \leqslant \sup_{x_0} \sup_{f[t]} \inf_{x[t]} \gamma(\vartheta \mid t_0, \Gamma[t_0], U, f)$$
$$(x_0 \in \Gamma[t_0], \ \varkappa[f[\tau], t] \leqslant \nu, \ \tau \leqslant t, t_0 \leqslant t \leqslant \vartheta) \tag{1.8}$$

for every possible initial estimate of $\Gamma[t_0]$.

(Uniqueness of the motions $x[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ corresonding to the initial state $x[t_0] = x_0$ is not assumed. This explains the symbol $\inf_{x[t]}$ in the right side of inequality (1.8). In the left side of (1.8) we mean any motion $x[t]$ which satisfies the indicated conditions.)

2. The tracking operation. The content of Problem 1.1 depends on the choice of the operations $\varphi^\circ[\{w, u\}, t]$ which determine the estimates of the domains $\Gamma[t]$. It is, of course, possible to pose the general problem of choosing the best operations φ° from some broad class. We could limit ourselves to a fairly narrow class of such operations, stipulating, however, that these must be more or less effectively describable. Our method of constructing these operations is as follows.

By the conditions of the problem we have a prior estimate of the domain Γ [0] of possible initial phase states x [0]. It is more convenient, however, to deal with the domain $G_0[0]$ rather than with Γ [0]; the former is the mapping of the domain Γ [0] into the m-dimensional space $\{q\}$ in accordance with the equation

$$q = \{X[\vartheta, 0]\, x\}_m \qquad (x \in \Gamma[0]) \tag{2.1}$$

where $X[t, t_0]$ is the fundamental matrix of solutions of the equation

$$dx/dt = A(t)\, x \tag{2.2}$$

It is clear that $G_0[0]$ is the totality of those points $q = \{x(\vartheta)\}_m$ to which system (2.2) is brought from the states $x(0) \in \Gamma[0]$ by the instant ϑ. We assume from now on that the domain $G_0[0]$ is a parallelepiped,

$$\alpha_i^\circ \leqslant q_i \leqslant \beta_i^\circ \qquad (i = 1, \ldots, m) \tag{2.3}$$

(If this is not valid for the initial domain Γ [0], then we can place Γ [0] in a larger domain Γ_* [0] for which the required condition is, in fact, fulfilled. The above condition also assists computation.)

Let the signal $w[\tau]$ (1.4) be realized by the instant t from the interval $0 \leqslant \tau \leqslant t$; let us assume, moreover, that the control $u[\tau]$ was acting over the interval $0 \leqslant \tau < t$. We are to construct an operator $\varphi[\{w[\tau], u[\tau]\}, t]$ which will reconstruct the vector

$$q = \{X[\vartheta, t]\, x[t]\}_m \tag{2.4}$$

from the signal $\{w[\tau], u[\tau]\}$ optimally in the sense that each of the coordinates q_i of the vector q is determined with the smallest possible error $\omega_i[t]$. (The construction of such optimal tracking operations is described in [5], pp. 279-291. We shall assume that the conditions of solvability of the problem laid down in this monograph are fulfilled). Thus, let the operation $\varphi[\{w, u\}, t]$ yield the result $q = q[t]$. This means that the state $x[t]$ which is actually realized at the instant t is such that from this state system (2.2) can, by the instant ϑ, pass only into a state $\{x(\vartheta)\}_m = q$ from some domain $G[t]$ described by the inequalities

$$|\, q_i - q_i[t] \,| \leqslant \omega_i[t] \qquad (i=1, \ldots, m) \tag{2.5}$$

These inequalities already provide an estimate of the domain $\Gamma[t]$. However, we must also take account of the results of previous estimates of the domains $G[t_*]$ for $t_* < t$, as well as the prior estimate of G [0] (2.3). We can do this in the following way. First, we assume that no estimates of the domain $\Gamma[t_*]$ for $0 < t_* < t$ were made in the segment $[0, t]$, and that by the instant t we have only the prior estimate of Γ [0]. Since the control $u[\tau]$ was operating on the segment $0 \leqslant \tau < t$, the domain Γ [0] known to us a priori is transformable into some domain $\Gamma[0, t]$. Its transformation is described by the linear transform

$$x_* = X[t, 0]\, x + \int_0^t X[t, \tau]\, B(\tau)\, u(\tau)\, d\tau \tag{2.6}$$

$$(x_* \in \Gamma[0, t],\; x \in \Gamma[0])$$

The domain $\Gamma[0, t]$ can once again be estimated conveniently in terms of its image $G_0[0, t]$ in the space $\{q\}$ under the linear transformation

$$q_* = \{X[\vartheta, t]\, x_*\}_m, \qquad x_* \in \Gamma[0, t] \tag{2.7}$$

Expressions (2.1) and (2.7) imply that the domain $G_0[0, t]$ is the domain $G\,[0]$ displaced by the vector

$$\Delta^u(0, t) = \left\{\int_0^t X[\vartheta, \tau]\, B(\tau)\, u(\tau)\, d\tau\right\}_m$$

Our prior estimate of the domain $\Gamma\,[0]$ therefore implies that by the moment t we can only have a phase state $x\,[t]$ such that system (2.2) can pass only into a state $\{x(\vartheta)\}_m = q$ from the domain $G_0[0, t]$ by the instant ϑ. But, recalling our description of the domain $G\,[t]$ (see [5], p. 338) and taking account of the result of the operation $\varphi[\{w, u\}, t]$, we can say that the state $x\,[t]$ realized at the instant t is such that from it system (2.2) necessarily passes into a state $q = \{x(\vartheta)\}_m$ lying in the intersection of the domains $G_0[0, t]$ and $G\,[t]$. Reasoning in similar fashion and recalling the results $G\,[t_*]$ of the operations $\varphi\,[\{w, u\}, t_*]$ for all $t_* \leqslant t$, we conclude that the state realized at the instant t is such that system (2.2) can only pass into a state $q = \{x(\vartheta)\}_m$ lying in the domain

$$G^\circ[t] = \bigcap_{0 \leqslant t_* \leqslant t} G\,[t_*, t] \cap G_0[0,t] \tag{2.8}$$

where $G\,[t_*, t]$ is the domain in the space $\{q\}$ obtainable from the domain $G\,[t_*]$ by displacement by the vector

$$\Delta^u(t_*, t) = \left\{\int_{t_*}^t X[\vartheta, t_*]\, B(\tau)\, u(\tau)\, d\tau\right\}_m \tag{2.9}$$

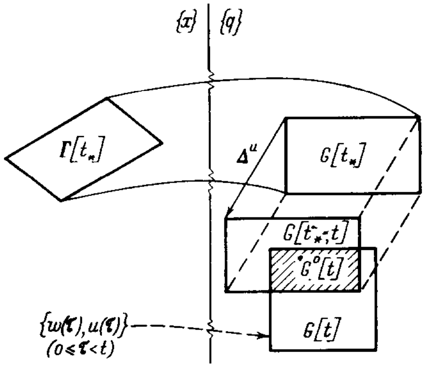

Fig. 1

It is this construction of the domain $G^\circ\,[t]$ which we call "the operation $\varphi^\circ\,[\{w, u\}, t]$." The domains $G^\circ\,[t]$ provide the required description of the domains $\Gamma\,[t]$. Hence, from now on we shall associate the sets $U\,(t, \Gamma\,[t])$ defining the strategy U with the domains $G^\circ\,[t]$, i.e. we shall specify contingencies (1.7) in the form

$$u\,[t] \in U\,(t, G^\circ\,[t]) \tag{2.10}$$

Finally, we note that the domains $G^\circ[t]$ are parallelepipeds, i.e. that

$$\alpha_i\,[t] \leqslant q_i \leqslant \beta_i\,[t] \qquad (i = 1, \ldots n) \tag{2.11}$$

This follows from the fact that all of the domains $G\,[t_*, t]$, which by (2.8) define the domain $G^\circ\,[t]$ (Fig. 1), are themselves papallelepipeds.

3. The correction problem as a convergence game. Problem 1.1 can be interpreted as a game of finding the minimax of the disparity between the coordinates of two controlled motions. Let us explain this. Let us assume that the domain $G^\circ\,[t]$(2.8) became known at the instant t and that the unknown state $x\,[t]$ was realized at the same instant. The symbol $x_*\,[t]$ denotes a phase state from the domain $\Gamma\,[t]$ which satisfies the condition

$$\{X[\vartheta, t]\; x_*\,[t]\}_m = q_*\,[t] \tag{3.1}$$

where $q_*\,[t]$ is the center of the domain $G^\circ\,[t]$, i.e.

$$q_{*i}\,[t] = {}^1/_2\,(\beta_i\,[t] + \alpha_i\,[t]) \qquad (i = 1, \ldots, m) \tag{3.2}$$

Let us write the equation of motion for the realization $x_*\,[t]$ in the form

$$dx_*/dt = A\ (t)\ x_* + B\ (t)\ u - v \tag{3.3}$$

where v is some fictitious control. We assume that the known state $x = x_*[t]$ is realized at the instant t in system (1.1) under consideration; we also assume that the "control" v can shift this state in an unknown fashion without taking it out of the domain $\Gamma\ [\tau]\ (\tau \geqslant t)$ for $\tau + 0$. (For example, to obtain the actually realized state $x\ [t] = x_*\ [t+0]$ we need merely set

$$v\ [\tau] = (x_*[t] - x\ [t])\ \delta\ (\tau - t)$$

where $\delta\ (\tau)$ is a delta function). Next, we express the vector $x_*\ [t]$ as a difference,

$$x_*\ [t] = y\ [t] - z\ [t] \tag{3.4}$$

and assume that the motion $y\ [t]$ is given by the equation

$$dy/dt = A\ (t)\ y + B\ (t)\ u \tag{3.5}$$

and the motion $z\ [t]$ by the equation

$$dz/dt = A\ (t)\ z + v \tag{3.6}$$

If the disparity between the motions $y\ [t]$ and $z\ [t]$ is defined as the quantity

$$\gamma_*\ [t] = \| \{y\ [t] - z\ [t+0]\}_m \| \tag{3.7}$$

then initial Problem 1.1 can be interpreted as one of choosing the optimal control u which provides the "pursuer" $y\ [t]$ (3.5) with the best possible convergence $\gamma_*\ [\vartheta]$ with the "target" $z\ [t]$ (3.6) for the least favorable behavior of the latter. The control u is restricted by condition (1.2), and the control v by the condition that the changes in the vector $z\ [t]$ which it produces always retain the vector $x_*\ [t+0] = y\ [t] - z\ [t+0]$ in the domains $\Gamma\ [t]\ (0 \leqslant t \leqslant \vartheta)$. Thus, the initial problem does, in fact, reduce to a certain games problem on the convergence of two motions.

4. The extremal construction. Let us solve the minimax problem for the quantity $\gamma_*\ [\vartheta]$ (3.7) on the basis of the extremal aiming rule [5, 6]. To this end we must consider the attainability domains (defined, for example, in [5], p. 331) $G^{(1)}\ (t, \vartheta, y)$ and $G^{(2)}\ (t, \vartheta, z)$ for the motions y and z.

The attainability domain $G^{(1)}\ (t, \vartheta, y)$ for the motion y (3.5) from the state $y\ [t] = y$ by the instant ϑ under restriction (1.2) is constructed in similar fashion and is described by the inequality (e.g. see [5])

$$\rho^{(1)}\ (t,\ l) + l'\ \{X\ [\vartheta,\ t]\ y\}_m - l'q \geqslant 0 \tag{4.1}$$

which is satisfied by the points q from the domain $G^{(1)}\ (t, \vartheta, y)$ (and only by these points) for all values of the m-dimensional vector l. Here

$$\rho^{(1)}\ (t,\ l) = \max_{u(\tau)\in U} \left[l' \left\{ \int_t^{\vartheta} X\ [\vartheta,\ \tau]\ B\ (\tau)\ u\ (\tau)\ d\tau \right\}_m \right] \tag{4.2}$$

where the prime indicates transposition.

Construction of the domains $G^{(2)}\ (t, \vartheta, z)$ is determined by the domains $G^\circ\ [t]$ (2.8), (2.11). In fact, let the state $z = z\ [t]$ be realized. By (3.4), $z\ [t] = y\ [t] - x_*\ [t]$. If the control $v\ (\tau)$ were equal to zero in the interval $t \leqslant \tau \leqslant \vartheta$, then system (3.6) would be in the state (see (3.1))

$$\begin{aligned}\{z\ (\vartheta)\}_m = q = \{X\ [\vartheta,\ t]\ (y[t] - x_*[t])\}_m = \\ = \{X\ [\vartheta,\ t]\ y\ [t]\}_m - q_*[t]\end{aligned} \tag{4.3}$$

by the instant ϑ.

However, under the action of the control $v(\tau)$ system (3.6) (in the sense of this control $v(\tau)$) can find itself in any state $\{z(\vartheta+0)\}_m = q$ which satisfies the inequalities

$$|q_i - \{X[\vartheta, t]\, y\,[t]\}_{mi} + q_{*i}\,[t]| \leqslant {}^1/_2\,(\beta_i\,[t] - \alpha_i\,[t]) \qquad (4.4)$$
$$(i = 1, \ldots, m)$$

It is these inequalities which define the domain $G^{(2)}(t, \vartheta, z\,[t])$. It is important to note that the estimates (4.4) realized in the course of the process determine the domains $G^{(2)}(t, \vartheta, z\,[t])$ in such a way that the basic conditions

$$G^{(2)}(t_*, \vartheta, z\,[t_*]) \supset G^{(2)}(t, \vartheta, z\,[t]) \quad (t_* < t) \qquad (4.5)$$

$$\{x\,[\vartheta+0]\}_m = \{y[\vartheta]\}_m - \{z\,[\vartheta+0]\}_m, \quad \{z\,[\vartheta+0]\}_m \in G^{(2)}(\vartheta, \vartheta, z\,[\vartheta]) \qquad (4.6)$$

are fulfilled.

Relations (4.5) and (4.6) enable us to interpret $G^{(2)}$ as the attainability domain for the motion z. This domain conforms to the argument of [5, 6] concerning extremal control schemes.

The validity of (4.6) follows directly from the definition of the domain $G^{(2)}$ (4.4) in the sense of the quantities $\alpha_i\,[t]$, $\beta_i\,[t]$ (2.11) and the vector $q_*\,[t]$ (3.1). Let us verify the validity of imbedding (4.5). We infer from (4.4) that $G^{(2)}(t_*, \vartheta, z\,[t_*])$ is the domain symmetric to the domain $G^\circ\,[t_*]$ with respect to the point $q = 0$ and displaced translationally by the vector

$$p\,[t_*] = \{X\,[\vartheta, t_*]\; y\,[t_*]\}_m \qquad (4.7)$$

while $G^{(2)}(t, \vartheta, z\,[t])$ $(t > t_*)$ is a domain symmetric to the domain $G^\circ[t]$ with respect to the point $q = 0$ and displaced translationally by the vector

$$p\,[t] = \{X\,[\vartheta, t]\, y\,[t]\}_m = \Big\{X\,[\vartheta, t_*]\, y\,[t_*] + \int_{t_*}^{t} X\,[\vartheta, \tau]\, B(\tau)\, u(\tau)\, d\tau\Big\}_m \qquad (4.8)$$

But according to (2.8), for $t > t_*$ the domain $G^\circ\,[t]$ forms a part of the domain $G^\circ\,[t_*]$ displaced by the vector $\Delta^u(t_*, t)$ (2.9). By virtue of (4.7), (4.8), this implies the validity of relation (4.5).

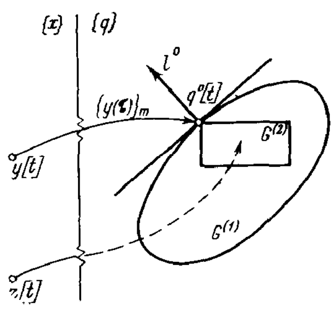

Fig. 2

Now let us describe an extremal aiming scheme appropriate to this case. Let the values of the motions $y\,[t]$ and $z\,[t]$ realized at the instant t be contained in the known parameters $q_*[t]$ (3.1) and $\alpha_i\,[t]$, $\beta_i\,[t]$ (2.11) from the domain $G^\circ\,[t]$. Assuming provisionally that the realized vectors $y\,[t]$ and $z\,[t]$ are known, we construct the attainability domain $G^{(2)}(t, \vartheta, z[t])$ and the smallest ε-neighborhood $G^{(1)}(t, \vartheta, y\,[t], \varepsilon^\circ)$ of the attainability domain $G^{(1)}(t, \vartheta, y\,[t])$ which contains the former attainability domain. If it turns out for $0 \leqslant t < \vartheta$ that whenever $\varepsilon^\circ > 0$, the boundaries of these domains intersect only along the sets $Q^\circ\,[t]$, each of which lies entirely in the hyperplane

$$l^{\circ\prime}\,[t]\, q = \rho\,[t] \qquad (4.9)$$

unique for each t, then we have the coarse case [6]. This means that the optimal control $u = u\,[t]$ which solves the problem consists of the condition of aiming of the motion $y\,[t]$ at any point $q^\circ\,[t] \in Q^\circ\,[t]$ (Fig. 2).

It is important to note that this control is determined by the known quantities $\alpha_i\,[t]$,

$\beta_i\ [t]$. In fact, the aiming control $u\ [t] = u_* \in U$ is here determined by the maximum condition $l^{\circ\prime}\{X[\vartheta, t]B(t)u_*\}_m = \max\limits_{u\in U}[l^{\circ\prime}\{X[\vartheta, t]B(t)u\}_m]$ (4.10)
where the m-dimensional unit vector l° is the exterior normal to the domain $G^{(1)}(t, \vartheta, y\ [t], \varepsilon^\circ\ [t])$ at the point $q^\circ\ [t]$ where the boundary of this domain comes in contact with the domain $G^{(2)}(t, \vartheta, z\ [t])$. But this vector $l^\circ\ [t]$ is determined solely by the relative positions of these domains. Because of the monotype character of systems (3.5), (3.6) and by virtue of the equation $x_*\ [t] = y\ [t] - z\ [t]$, this relative disposition of the domains $G^{(1)}$ and $G^{(2)}$ is determined solely by the parameters $\alpha_i\ [t]$, $\beta_i\ [t]$, and $\rho^{(1)}$ of the domains $G^{(1)}$ and $G^{(2)}$, which therefore determine the vector l° and with it the optimal control $u^\circ\ [t] = u_*$ (4.10). (The validity of this statement is also evident directly from relations (4.1), (4.4) which define the domains $G^{(1)}$ and $G^{(2)}$.)

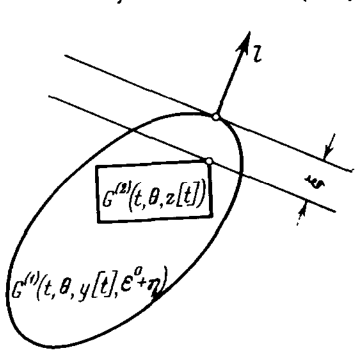

Fig. 3

Thus, in accordance with [6], Problem 1.1 has a solution in the coarse case. The strategy U° is given by the sets $U^\circ(t, \alpha_1\ [t], \ldots, \alpha_m\ [t], \beta_1\ [t], \ldots, \beta_m\ [t])$, consisting of all the vectors $u_* \in U$ which satisfy conditions (4.10). This strategy ensures a correction result $\gamma\ [\vartheta]$ not worse than $\varepsilon^\circ\ [0]$, i.e. $\gamma\ [\vartheta] \leqslant \varepsilon^\circ\ [0]$ (4.11)

It should be noted that the coarse case for the above games problem of convergence derived from initial correction Problem 1.1 is rather exceptional. The next section is therefore concerned with the more natural regularizable critical case.

5. The critical case. Let a situation be realized in which by a certain instant $t = t_0 \geqslant 0$ the boundary $H^{(1)}(t_0, \vartheta, y\ [t_0]; \varepsilon^\circ\ [t_0])$ $(\varepsilon^\circ > 0)$ of the domain $G^{(1)}(t_0, \vartheta, y\ [t_0]; \varepsilon^\circ\ [t_0])$ has an intersection $Q^\circ\ [t_0]$ with the domain $G^{(2)}(t_0, \vartheta, z[t_0])$ which does not fit into a unique hyperplane of the form (4.9). Let us construct the domain $G^{(1)}(t_0, \vartheta, y\ [t_0]; \varepsilon^\circ\ [t_0] + \eta)$, where $\eta > 0$ is a small constant. From the definition of the quantity ε° we infer that the domain $G^{(2)}(t_0, \vartheta, z\ [t_0])$ lies strictly within the domain $G^{(1)}(t_0, \vartheta, y\ [t_0]; \varepsilon^\circ\ [t_0] + \eta)$. Our choice of the control $u\ [t]$ for $t \geqslant t_0$ is subject to the desire to preserve the imbedding

$$G^{(2)}(t, \vartheta, z\ [t]; \varepsilon\ [t]) \subset G^{(1)}(t, \vartheta, y\ [t]; \varepsilon^\circ\ [t_0] + \eta\ [t]) \qquad (5.1)$$

for $\varepsilon\ [t] > 0$ and for $\eta\ [t] = \eta = \text{const}$, or, at the very least, under the condition of maximally slow increase of the variable $\eta\ [t]$ with increasing time t. We construct the control u in the following way.

Let a state $\{t_*, y\ [t_*] = y_*, z\ [t_*] = z_*\}$ (characterized by the measurable quantities $q_*[t_*]$, $\alpha_i\ [t_*]$, $\beta_i\ [t_*]$ $(i = 1, \ldots, m)$) be realized by some instant $t = t_* \geqslant t_0$. We assume that this is accompanied by the fulfilment of condition (5.1) with some value $\eta\ [t_*]$ realized by the given instant $t = t_*$. Let us choose some m-vector l of the exterior normal to the domain $G^{(1)}(t_*, \vartheta, y_*; \varepsilon^\circ\ [t_0] + \eta\ [t_*])$. The symbol $\xi(l, t_*, y_*, G^{(2)}(t_*, \vartheta, z_*)) = \xi(l, t_*\{\alpha_i[t_*], \beta_i[t_*]\})$ denotes the distance between the hyperplanes $L^{(1)}(l)$ and $L^{(2)}(l)$ which are orthogonal to l and tangent to the domains $G^{(1)}$ and $G^{(2)}$ (Fig. 3).

Let us construct the function

$$\lambda(t, y, G^{(2)}(t, \vartheta, z)) = \int\limits_{\|l\| \leqslant 1} \varphi[\zeta(l, t, y, G^{(2)})]\, d\zeta \tag{5.2}$$

where $\varphi[\xi] \geqslant 0$ is a smooth function monotonic for $\xi > 0$ and of order ξ^{-n} as $\xi \to 0$; the integral is taken over the sphere $\|l\| \leqslant 1$. Let us assume that for $t_* \leqslant t \leqslant t_* + \Delta t$ the quantity $y(t)$ varies in accordance with Eq. (3.5), where $u = u(t)$ is some measurable function continuous from the right at the point $t = t_*$; let the domain $G^{(2)}(t, \vartheta, z(t))$ vary in accordance with the measurements of the signal $w(\tau)$ (see Sect. 4); we assume that the quantity $\eta(t)$ is constant on the segment $[t_*, t_* + \Delta t]$. Next, considering the variation of the function $\lambda(t) = \lambda(t, y(t), G^{(2)}(t, \vartheta, z(t))$ in the small interval $[t_*, t_* + \Delta t]$, we can use the standard reasoning of maximum-principle theory [7] to obtain the estimate

$$\left(\frac{d\lambda}{dt}\right)_+^{(b)} \leqslant -s'(t_*, G^\circ[t_*])\, B(t_*)\, u(t_*) + \chi \tag{5.3}$$

characterizing the effect of the control u on the variation of the function $\lambda(t)$.

Here $(d\lambda / dt)_+^{(b)}$ denotes the right-hand upper derivative of the function $\lambda(t)$ for $t = t_*$; the n-dimensional vector s is defined by the equation

$$s(t, G^\circ[t]) = \int\limits_{\|l\| \leqslant 1} \frac{d\varphi}{d\xi} \psi(t_*, l)\, d\zeta \tag{5.4}$$

where the n-dimensional vector function $\varphi(t, l)$ is the solution of the equation

$$\frac{d\psi}{dt} = -A'(t)\psi \tag{5.5}$$

under the boundary condition

$$\psi'(\vartheta, l)) = \{l', 0\}$$

The term χ appearing in the right side of (5.3) will not be involved in the subsequent construction of the control $u[t_*]$. Knowledge of the expression for χ is necessary in estimating the quantity $(d\lambda / dt_+^{(b)})$. It is not possible to develop here a detailed prior estimate for this quantity which would be effective in a more or less general case. We shall therefore omit further consideration of the term χ.

The first term in the right side of (5.3) reaches its minimum under a control $u(t_*) = u_* \in U$ which satisfies the following randomized maximum condition:

$$s'(t_*, G^\circ[t_*])\, B(t_*)\, u_* = \max_{u \in U} s'(t_*, G^\circ[t_*])\, B(t_*)\, u \tag{5.6}$$

The strategy U° which determines the control u will be defined in terms of the sets $U^\circ(t_*, G^\circ[t_*])$ consisting of all those vectors u_* from U which satisfy condition (5.6). Under sufficiently broad conditions this strategy U° is permissible; moreover, an estimate of the form (5.3) is valid along the true realizations of the motions $y[t]$, $z[t]$ for $u[t_*] \in U^\circ(t_*, G^\circ[t_*])$. If the inequality

$$\left(\frac{d\lambda}{dt}\right)_+^{(b)} \leqslant -s'(t, G^\circ[t])\, B(t)\, u[t] + \chi < \mu \tag{5.7}$$

$$(u[t] \in U^\circ(t, G^\circ[t]), \mu = \text{const})$$

is valid for $\eta[t] = \eta[t_0] = \text{const}$ for all times $t \geqslant t_0$, then the realization $\lambda[t]$ remains bounded for all $t \geqslant t_0$. However, by the construction of the function λ (5.2), this implies that imbedding (5.1) for $\eta[t] = \eta[t_0]$ is valid for all $t \geqslant t_0$. But for $t = \vartheta$ this imbedding implies that

$$\gamma[\vartheta] \leqslant \varepsilon^\circ[t_0] + \eta[t_0] \tag{5.8}$$

We call the case where condition (5.7) is fulfilled the "regularizable critical case".

Thus, in the regularizable critical case we construct a strategy U° which yields a correction result γ $[\vartheta]$ not worse than (5.8).

If inequality (5.7) cannot be fulfilled, then we can make use of the same strategy U° described at the beginning of the present section. However, preservation of imbedding (5.1) then requires alteration of the quantity $\eta[t]$. When η $[t]$ changes, the right side of inequality (5.7) contains an additional term ν of the form

$$\nu = \left(\frac{d\eta}{dt}\right)_{(H)+} \int\limits_{\|l\| \leqslant 1} \frac{d\varphi}{d\xi}\, d\zeta \tag{5.9}$$

and the value of the lower right-hand derivative $(d\eta\ /\ dt)_{(H)+}$ can be chosen such that, for example, $(d\lambda\ /\ dt)_+^{(b)} \leqslant 0$. If estimation of the required value of $(d\eta\ /\ dt)_{(H)+}$ from the realized position $\{t,\ G\ [t]\}$ is not easy because of the difficulties of prior estimation of χ, then it is possible to choose the values of $(d\eta\ /\ dt)_{(H)+}$ at each instant t, e.g. from the conditions

$$\left(\frac{d\lambda}{dt}\right)_+^{(b)} + \left(\frac{d\lambda}{dt}\right)_{(H)-} \leqslant 0 \tag{5.10}$$

where the lower left-hand derivative $(d\lambda/\ dt)_{(H)-}$ of the function is determined from a past realization of $\lambda\ (t)$ which has become known. (The control method actually specified must, of course, be implemented by a discrete scheme). It should be noted that the control method just described has to be evaluated for stability, since it involves a differentiation operation.

6. Comparison with the minimax strategy. A correction problem similar to that with which we began our formulation of Problem 1.1 is considered in [4]. However, the method of correction described in [4] differs from the control method described here with reference to Problem 1.1 (Shelement'ev [4] describes a discrete-time scheme for allowing for the measurements of the signal $w\ (\tau)$; but this is not important, since the same scheme can be extended to the continuous case). The essential difference between the control methods is as follows. Employing the language of systems (3.5), (3.6), we can say that the control $u\ [t]$ is chosen in [4] at the instant t from the condition of aiming of the motion $y\ [t]$ at the point $q = q_0$ from the attainability $G^{(1)}(t,\ \vartheta,\ y\ [t])$ in which the minimax

$$\min_q \max_p \| p - q \| = \varepsilon_0\ [t] \tag{6.1}$$

is attained for all q from $G^{(1)}(t,\ \vartheta,\ y\ [t])$ and all p from $G^{(2)}(t,\ \vartheta,\ z\ [t])$. Fulfilment of the correction result

$$\gamma\ [\vartheta] \leqslant \varepsilon_0\ [t] \tag{6.2}$$

is ensured.

In contrast to this, we choose the control $u\ [t]$ on the basis of the condition of aiming of the motion $y\ [t]$ at a point $q = q_0$ from the domain $G^{(1)}(t,\ \vartheta,\ y\ [t])$ which corresponds to the maximin

$$\max_p \min_q \| p - q \| = \varepsilon^\circ\ [t] \tag{6.3}$$

for all q from $G^{(1)}(t,\ \vartheta,\ y\ [t])$ and for all p from $G^{(2)}(t,\ \vartheta,\ z\ [t])$. Since $\varepsilon^\circ\ [t] \leqslant \varepsilon_0 [t]$, it follows that the latter control method must generally yield a better result in the coarse case, or at least in the regularizable case, i.e. when inequalities (4.11) or (5.8) are fulfilled. We must bear in mind, however, that inequality (6.2) is always fulfilled, while inequalities (4.11) or (5.8) are fulfilled only under the stated conditions.

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ONE FORM OF THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS

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Displacement operators constructed with the aid of all the constraints are used to derive a form of the equations of motion which is valid for both holonomic and nonholonomic mechanical systems. In the case of holonomic systems the equations coincide with the familiar equations of Poincaré [1, 2].

1. Constructing the displacement operators. Let the positions of a mechanical system with l degrees of freedom be defined by the n variables $x_1, \ldots$ $\ldots, x_n$ subject to $n - l$ linear constraints

$$\eta_j\, dt \equiv \sum_{i=1}^{n} a_{ji}\, dx_i + a_{j0}\, dt = 0 \qquad (j = l+1, \ldots, n) \tag{1.1}$$

on the real displacenets, and to the equations

$$\omega_j \equiv \sum_{i=1}^{n} a_{ji} \delta x_i = 0 \qquad (j = l+1, \ldots, n) \tag{1.2}$$

on the virtual displacements.

Here a_{ji}, a_{j0} are functions of the variables t, x_i; dx_i, δx_i are the differentials and variations of the variables x_i on the real and virtual displacements of the system.

Following Chetaev [2], we complement (1.2) by a system of l linear differential forms

$$\omega_1, \ldots, \omega_l \tag{1.3}$$

which are independent of each other and also with respect to the forms $\omega_{l+1}, \ldots, \omega_n$ of (1.2). Next, we define the total variation of the function $f(t, x_i)$ by the formula